DYNAMICAL MODELING OF A TORNADO

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Introduction

"The dark side of nature."
-Tagline to the 1996 film ***Twister***

As one of the most spectacular and destructive phenomena in meteorology, tornadoes are a fascinating object for physical study. Tornadoes are violently twisting upward vortices of air produced by large storm systems commonly known as supercells [1]. Tornadoes can occur anywhere in the world, but the epicenter of tornado activity is in the central United States, known as 'tornado alley', which has favorable conditions for tornado formation. The climate conditions there which engender tornado production are the intersection of warm, moist air from the Gulf of Mexico, and cold, dry air from the Rocky Mountains and Canada (see Figure A.2).

In this paper, a simple model of a tornado will be developed which will follow and showcase basic principles of fluid mechanics. Some defining behaviors of a tornado which this model will attempt to describe are outlined in the following objectives:

- Description of a mesocyclone, in which a large column of air within a cloud begins to rotate.
- Upward lift from the rotating column of air.
- Formation of a funnel cloud with a dynamic radius.
- Formation of multiple vortices from the same cloud.

To this end, we will define a simple set of equations from the principles of classical mechanics in order to model the above phenomena. The focus will be on the derivations from first principles and general progression to a specific model, and then the explanations and predictions which come from the model.

Chapter 1

Fluid Dynamics

> *"When I meet God, I am going to ask him two questions: Why relativity? And why turbulence? I really believe he will have an answer for the first."*
> *-Werner Heisenberg*

Fluid dynamics presents a remarkable intellectual challenge due to the difficulty of finding closed-form analytical solutions, the enormous complexity of the systems involved, and the tendency of simulations to choke even the most formidable computers.

Tornadoes are also difficult to study due to the problems with directly observing data and gathering complete data sets. Consequently, much of our knowledge of tornadoes comes from extensive study of numerical simulations or simplified vortex chamber models.

We will begin with a treatment of basic fluid dynamics and then will proceed with the application to tornadoes by applying the principles described in this chapter to the Ward chamber, a successful and influential tornado model [2, 3, 4]. The derivation in this section is taken from [5], with steps and notation slightly modified for the purposes of this paper.

Motivating Questions

Like many topics in mathematics and physics, we will begin by pinning down things which intuition and reason tell us to be true and seeing what can be deduced from there. The motivating questions which will guide us to the Euler equations of fluid motion are:

- How does the density of the fluid change in time and space?
- How does the velocity of the fluid change in time and space?
- What is the relationship between external pressures and the density and velocity of the fluid?

Derivation from First Principles

We begin with a fairly sensible principle: that the total change of mass in a volume of consideration is equal to the total mass flowing into or out of the volume. For mass M, fluid density ρ, velocity $\mathbf{v}$, the volume V, and flow in or out of a surface S enclosing the volume over an infinitesimal time interval ∂t,

$$M = \int \rho dV \tag{1.1}$$

$$\rightarrow \frac{\partial M}{\partial t} = \int \frac{\partial \rho}{\partial t} dV = \oint \rho \mathbf{v} d\mathbf{S} = -\int \nabla \cdot (\rho \mathbf{v}) dV. \tag{1.2}$$

Further, the basic laws of Newtonian mechanics can be applied to a continuous medium, and we can safely postulate that in a 'control volume' in which the mass is conserved, momentum is conserved as well. Thus, by introducing a velocity into equation (1.1) our conservation of *mass* is recast as a familiar conservation of *momentum*.

We use Einstein notation, in which the index i is assumed to be summed over $i = 1, 2, 3$ corresponding to 3 spatial dimensions (such as x, y, and z). Where Π_i is the total momentum in a given direction which comes from the product of mass density ρ and velocity vector $\mathbf{v}$ with components v_i flowing through an infinitesimal volume dV enclosed by surface $\mathbf{dS}$,

$$\Pi_i = \int \rho v_i dV, \tag{1.3}$$

$$\frac{\partial \Pi_i}{\partial t} = \int \frac{\partial(\rho v_i)}{\partial t} dV = \oint (\rho v_i)\mathbf{v} \cdot \mathbf{dS}. \tag{1.4}$$

If we assume that no external pressure or forces are applied to the volume under consideration, then the only way the total summated momentum in the volume can change is by flow *in* or *out*. We express this through the language of integral calculus in (1.4).

We have made clear progress in answering our first two motivating questions concerning the density and velocity change over time by invoking intuitive conservation laws and familiar Newtonian principles. Let us now take steps in establishing the relationship of external pressures with density and velocity.

Momentum can clearly also change by the presence of forces in or onto the fluid, and not just through the introduction of moving matter to the volume under study. Pressures act to introduce momentum into a volume of consideration. In applications, this could come from a depressing cylinder, or electron degeneracy pressure acting in a neutron star. With the force of pressure in a given direction i as F_{ip}, we integrate this pressure P over components of a surface d_iS enclosing a volume dV:

$$F_{iP} = \oint P d_i S = -\int \nabla_i P dV. \tag{1.5}$$

Fields such as gravity also influence the motion of fluids. An abstract field is represented here by ϕ. Beyond gravity, electromagnetic fields are relevant to complex fluid systems such as plasmas, in which every particle species in the fluid is electrically charged and susceptible to field effects. Where ρ represents density and ϕ_i represents the field acceleration in the i direction, the force $F_{i\phi}$ in the i direction due to ϕ over an infinitesimal volume unit dV is

$$F_{i\phi} = \int \rho \phi_i dV. \tag{1.6}$$

The effects of these forces may be summarized in the following expression:

$$\frac{d\Pi_i}{dt} = F_{i\phi} + F_{iP} \tag{1.7}$$

We pause here to recapitulate, and note that we may combine a statement of conservation of momentum with a description of forces to yield equations of motion. The two left terms represent the change in momentum moving laterally in an i direction, and changing from the i to j direction respectively, and the two right terms represent the pressure acting on the system summed with the force imposed by field forces ϕ:

$$\frac{\partial(\rho v_i)}{\partial t} + \nabla_j(\rho v_i v_j) = -\nabla_i P + \rho\phi_i. \tag{1.8}$$

At this point, we are mostly done. All that remains is a brief simplification which is mostly mathematical manipulation. Rewriting the left term in vector notation (preserving the meaning of momentum conservation) and dividing by ρ delivers us **Euler's Equation**, which is given as:

$$\boxed{\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{-\nabla P}{\rho} + \phi} \tag{1.9}$$

This equation is critical to the study of fluid dynamics since it gives an exact description of the motion of fluid without viscosity. It will be the first equation we begin with in the following chapter as we apply the tools of fluid dynamics to understanding chaotic weather behavior.

Chapter 2

Tornado Dynamics

*"Twist and shout \\
C'mon, c'mon, c'mon, c'mon, baby, now \\
Come on and work it on out*
- Twist and Shout*, by Phil Medley, as made famous by The Beatles*

Armed with some principles of fluid dynamics, we are now prepared to evaluate tornado dynamics. The explanatory power of this model will be discussed in Chapter 3 after the mathematics and equations are developed in this chapter.

2.1 The Ward Vortex Chamber

Much like in a study of the two-body central force orbit problem, we must make many simplifying assumptions in constructing our model in order to make progress. The numerical model which we will describe in this paper is known as the Ward vortex chamber and will henceforth be referred to as the Ward model. The equations which govern the simulation were detailed in a 1972 paper [3] and have been extended since [4]. They have also been experimentally tested in a physical vortex chamber, the results of which will be documented at the end of this chapter and Chapter 3. Assumptions that are made in the model will be stated as we proceed.

We begin with a modified form of the Euler equation. This equation refers to the inward unit vector **n** normal to the surface S, the static pressure p, the fluid density ρ, external forces **F**, and velocity at a point **q**. The equation states that the pressure acting on the surface is equal to the change in momentum in the volume, minus the fluid density ρ times the external forces $\mathbf{F}$, and minus the momentum flowing in and out of the system through the surface.

$$\int_S \mathbf{n} p dS = \frac{\partial}{\partial t}\int_V \rho \mathbf{q} dV - \int_V \rho \mathbf{F} dV - \int_S (\mathbf{n}\cdot \mathbf{q})\rho \mathbf{q} dS \tag{2.1}$$

We now make some assumptions which will eliminate or simplify terms. The first is that the fluid density ρ is constant through space. Second, that the flow into a volume of consideration occurs at a constant rate, so the first term on the right side is zero. We lastly assume that the external body forces are negligible, which eliminates the second term. We are then left with the neater equation which we will use heavily:

$$\int_S \mathbf{n} p dS = -\rho \int_S (\mathbf{n}\cdot \mathbf{q})\mathbf{q} dS \tag{2.2}$$

Which states that the pressure applied onto an arbitrary surface S (by multiplying the vector pointing inwards, $\mathbf{n}$ by the pressure p) is equal to the momentum flowing into or out of the system through the same surface.

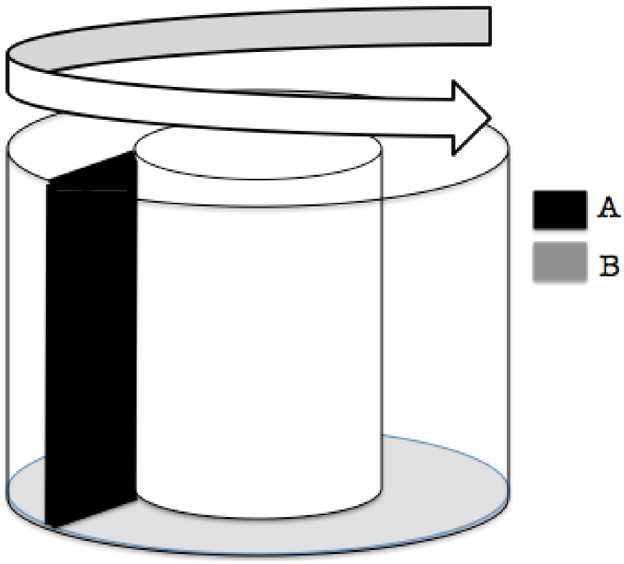

Figure 2.1: A rough depiction of the Ward model with radial plane A and cylindrical base plane B highlighted.

Now we continue in developing the Ward model and apply it to the cylindrical model. The model described is a cylinder with a small hole in the center detailed in Figure 2.1. We will apply Equation (2.2) to this and make reference to an arbitrary (r, z) plane A extending radially outward from the core and a (r, ϕ) plane B at the bottom of the cylinder to assist in understanding the geometry. We also consider a slice of a cylindrical wedge defined by an angle δ, and in the limit of small angles, we approximate all usage of $\sin\left(\frac{\delta}{2}\right)$ as $\frac{\delta}{2}$.

Two lengthy equations follow from (2.2). Because (2.2) concerns vectors on both sides, we can extract information about the radial and angular behavior of the fluid by equating components in the r, ϕ, and z direction.

2.2 Upwards Lift

We will derive equation (2.3), which describes momentum leaving the system upwards. We begin by equating the z terms in (2.2). In the following term, p_{top} and p_{bottom} represent the average pressure on the top and bottom of the system respectively. r_{inner} and r_{outer} represent the radius of the hole in the center of the cylinder (the vortex core) and of the cylinder itself respectively. δ represents the wedge angle in a wedge of the cylinder. The left side of the equation represents the difference of the average pressure times the area in a small wedge defined by δ. This is given clearer articulation on the right side of the equation, in which we see the integration of w_3^2 which represents the magnitude of the z-component of the velocity multiplied by the mass ρ. Note that we will henceforth denote the average values of variables as the variable symbol with a bar over the top. For example, $\bar{p}$ represents an average pressure.

$$(\bar{p}_{top} - \bar{p}_{bottom})(r_{outer}^2 - r_{inner}^2)\frac{\delta}{2} = \delta\rho \int_{rinner}^{router} w_3^2(r) r dr \tag{2.3}$$

What results is a statement that the *flow of upward momentum through the system is equal to the upward pressure*. This partially describes one of our goals from the beginning of this paper, which was to find an expression for the upward lift from a rotating column of air.

2.3 Radial Momentum Flow

We consider now the implications which come from equating the radial pressure thrust towards or away from the vortex core and the flow of momentum in the radial direction.

Define h to be the height of the cylinder, p_{vert} to be the pressure over a vertical section, p_{inner} to be the pressure on the inner cylindrical wall, p_{outer} to be the pressure on the outer cylindrical wall, v to be the velocity in the radial direction, and u to be the velocity in the ϕ direction.

$$\begin{aligned}\bar{p}_{vert}(r_{outer} - r_{inner})h\delta - (\bar{p}_{outer}r_{outer} - \bar{p}_{inner}r_{inner})h\delta = \\ \rho u^2 r_{outer}h\delta - \delta\rho \int_{rinner}^{router}\int_0^h v^2(\mathbf{r}, z)drdz - \\ \delta\rho \int_{rinner}^{router} w_3(r)u_3(r)rdr\end{aligned} \tag{2.4}$$

The physical interpretation of the terms is as follows: the leftmost term of equation (2.4) reflects the vertical pressure forces along the radial interval. The second term shows the difference of pressure forces on the inner and outer sides of the cylinder. On the right side, the first term is the flow of radial momentum through the center of the cylinder. The centrifugal effect is accounted for by the double integral term. The final integral represents upward drift of radial momentum.

The physical interpretation of the entire equation can be summarized as follows: *The pressure through a radial slice is equivalent to the flow of radial momentum through the slice.* The true utility of equation (2.4) comes from a series of approximations which will be omitted due to their tedium. We will instead focus on the predictions and explanations which the simplifications suggest.

The size of r_{inner}, the vortex core, can be approximated as a function of the size of the cylinder r_{outer}, and θ, the inflow angle of momentum. K is a constant.

$$r_{inner} = r_{outer}\frac{\sin^2\theta}{1 - K\sin^2\theta} \tag{2.5}$$

If inflow of momentum to the cylinder is purely in the radial direction and has no tangential component, then the tangential velocities vanish, the inner radius of the vortex core goes to zero, and (2.4) becomes

$$(p_{vert} - p_{outer})r_{outer}\delta h = \rho u^2 r_{outer}\delta h. \tag{2.6}$$

The seminal paper by Ward *et al* [3] investigated the influence of the inflow

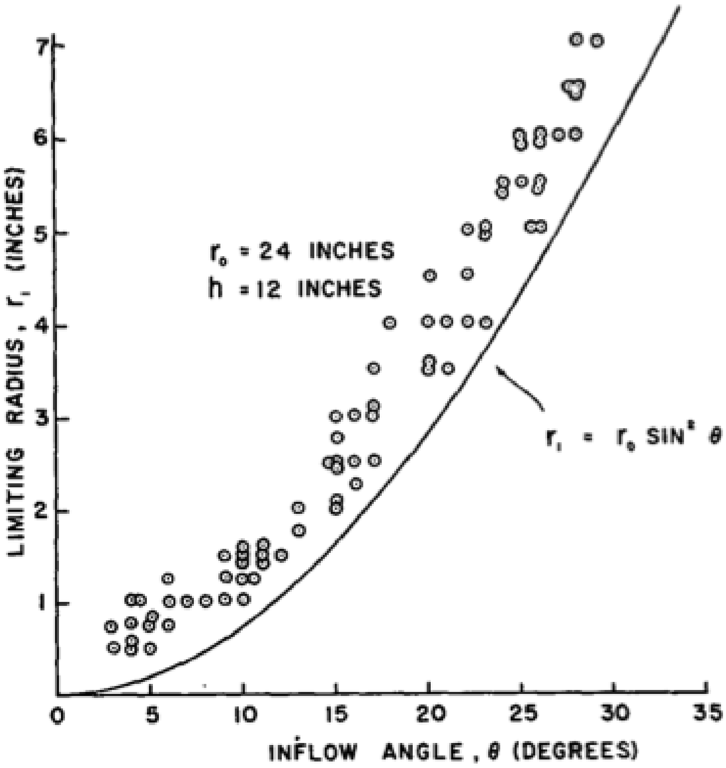

Figure 2.2: The numerical findings of [3] comparing the inflow angle to the radius of the vortex core.

angle on the diameter of the vortex core. Their numerical findings are detailed in Figure 2.2, and their experimental findings showing the contraction of the vortex core are found in Figure 2.3.

2.4 Swirl Ratio

Before proceeding, we will define a quantity S to be the swirl ratio. From the outside radius of the cylinder r_{outer}, the total angular momentum of the cylinder $2\pi L$, and the volume flow rate $2\pi Q$ (the total flux of fluid through the bottom of the system), we define

$$S = \frac{r_{outer} L}{2Q} \tag{2.7}$$

This simple non-dimensional parameter is correlated with certain vortex behaviors which are found in nature (see Figure A.1) and also observed experimentally (see Figure 3.3). Increasing angular momentum and vortex size

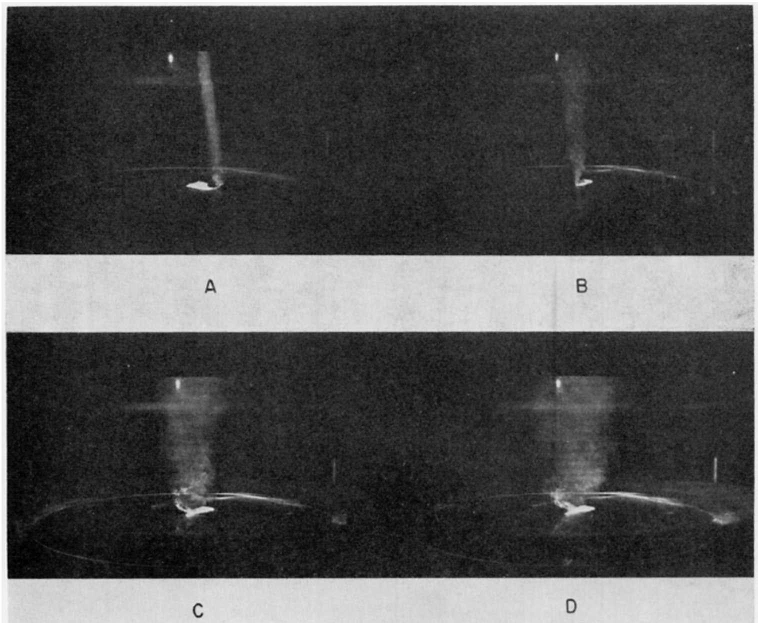

Figure 2.3: Demonstration of the contraction of the vortex core with increasing inflow angle from [3].

with a small volume flow rate through the bottom of the vortex was shown by Ward and others to support the formation of multiple vortices.

To briefly recapitulate, we began with a derivation of Euler's equation from first principles, and applied it to the case of a cylindrical fluid with a vortex core of variable size. We have arrived at a model of a rotating fluid (meeting our first objective of modeling a rotating column of air), with upward flow (our second objective) which can expand and contract dynamically based on the inflow of air (our third objective). The swirl ratio S will be shown in Chapter 3 to relate to the formation of multiple vortices (our fourth objective).

In the next chapter, we will describe interesting results in tornado modeling and physical understanding which follow directly from the equations described in this chapter.

Chapter 3

Additional Tornado Dynamics

"In theory, there's no difference between theory and practice, but in practice there is." -Yogi Berra

Several results, figures, and auxiliary studies which build off of the equations from the Ward model and the principles covered in Chapter 2 will be presented in this chapter.

Influx Angle: The Ward model showed that the influx of air and the angle at which it occurs are influential to the behavior of a tornado. In supercells, this influx is observed to come from intersecting temperature fronts [1]. Marquis *et al* [6] developed a hypothesis that a thermal boundary provides the influx and uplift within the tornado. Figure 3.1 shows the air influx into the tornado across temperature boundaries. It is hypothesized that when the air is too cold, the denser and heavier air sufficiently resists updraft to arrest the production of a tornado. Notice that in Figure 3.1, the thermal boundary intersects the

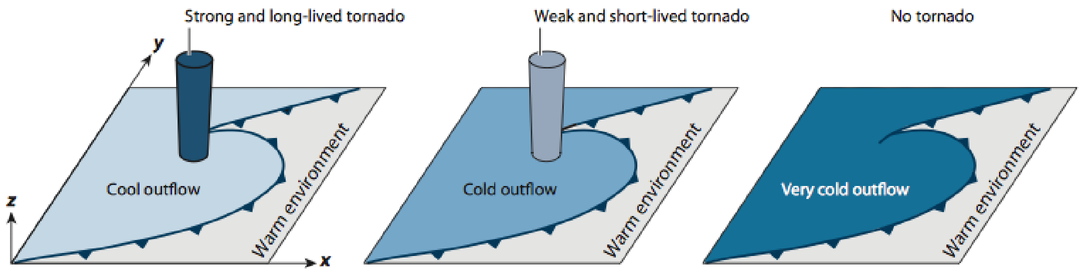

Figure 3.1: Visualization of temperature boundaries on the formation and strength of a tornado. Note that the air front intersects the tornado at an angle. Figure from [2], interpretation from [6].

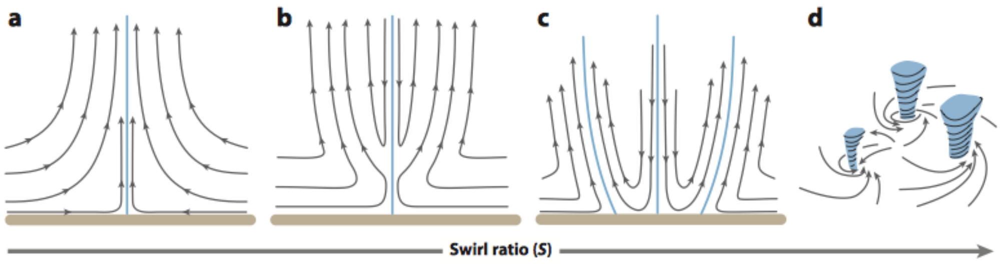

Figure 3.2: The formation of multiple vortex cores is depicted diagrammatically as the swirl ratio increases. Figure from [2], interpretation from [3].

tornado at an angle; as it was shown in [3] if the influx has an angle of zero, the tangential velocities vanish and the vortex core of the tornado disappears.

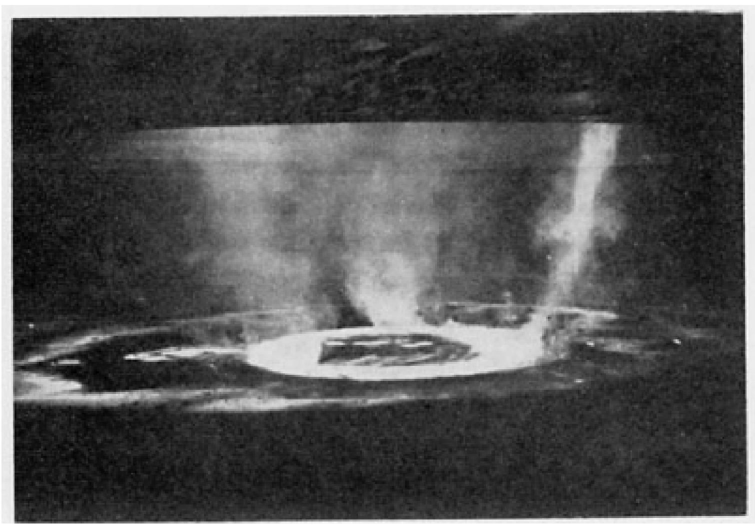

Figure 3.3: A photograph from [3] demonstrating the formation of multiple vortices in the Ward chamber. This result is produced when the swirl ratio S is increased.

Multiple Vortices: Another corollary of the Ward model was a theoretical link to a truly mystifying tornado behavior: the formation of multiple vortices. We now invoke the swirl ratio S as defined in (2.7): It was shown experimentally by [3] that the swirl ratio is correlated with the formation of multiple vortices. Their findings are seen experimentally in Figure 3.3 and diagrammatically in 3.2. The power of the Ward model lies not just in connecting the angle of influx, but also in providing clues to the reasons behind formation of multiple vortex cores.

Chapter 4

Conclusion

"The most incomprehensible thing about the universe
is that it is comprehensible."
-Albert Einstein

Tornadoes represent a fascinating object of study due to their intersection of fluid mechanics, chaotic behavior, and a spectacular real-world phenomenon. An awe-inspiring natural occurrence of staggering scale and complexity, what is perhaps more surprising than the fact they exist at all is that with relatively simple mathematical tools and carefully applied reasoning, their behavior can be understood and predicted.

In this paper, we briefly described tornadoes, the conditions of their production, and certain characteristic behaviors which a good model should emulate. We then derived Euler's equations of fluid mechanics from first principles in Chapter 1. These equations were then applied in Chapter 2 to the context of modeling tornados in the Ward chamber, which applies many simplifying assumptions to make the model accessible to analytic techniques. We then discussed in Chapter 3 how this model has both **explanatory** and **predictive** capabilities, and that by defining certain parameters such as swirl ratio or influx angle, we can recover observed behavior in nature for vortex formation and tornado shape.

Even without exhaustive computational simulations, we see that simple analytic models can recover characteristic tornado behaviors. The fact that these results follow so quickly from *a priori* assumptions combined with simplifying approximations speaks to the tremendous beauty and successes of the principles of physics.

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Appendix A

Supplementary Figures

"A picture is worth a thousand words."
- Unknown

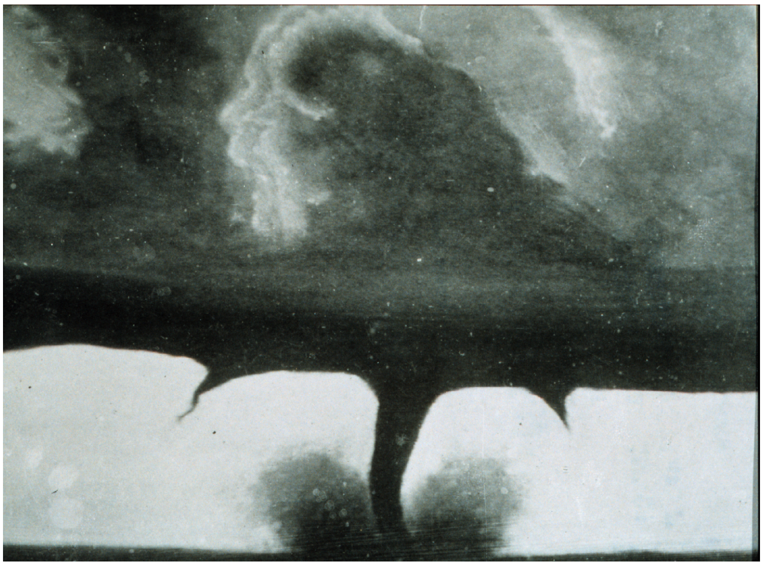

Figure A.1: The oldest known photograph of a tornado taken on August 28th, 1884. The formation of multiple vortices is evident, with two funnel clouds budding on the sides of the central vortex. [7].

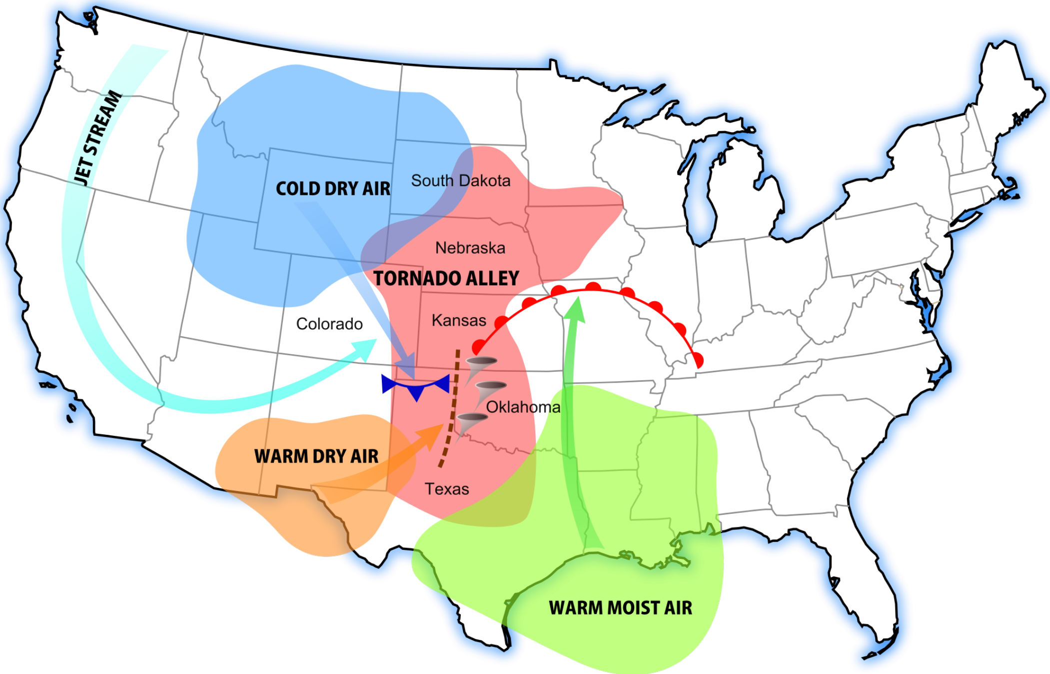

Figure A.2: Graphic demonstrating the confluence of factors which give tornado alley it's name. Note the intersection of radically different climatological air profiles. [8]

Fujita Scale		Enhanced Fujita Scale* * In use since 2007	
F–0	40–72 mph winds	EF–0	65–85 mph winds
F–1	73–112 mph	EF–1	86–110 mph
F–2	113–157 mph	EF–2	111–135 mph
F–3	158–206 mph	EF–3	136–165 mph
F–4	207–260 mph	EF–4	166–200 mph
F–5	261–318 mph	EF–5	>200 mph

Figure A.3: The Fujita scale, developed by tornado researcher Ted Fujita of the University of Chicago in 1971. This classification system describes the strength of tornadoes based on their average windspeed. [9]

Figure A.4: A visually striking “dust devil”, a classification of tornado characterized by weak strength and formation in a dry and arid region. [10]